Analysis of a Geo/Geo/1 Discrete-Time Inventory Model with Backlog using Matrix Analytic Method

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Abstract- This paper considers a discrete (s, S) inventory system with positive service and lead time. The arrival of customers constitutes a Bernoulli process.Service time and lead-times are geometrically distributed. The maximum storage level of inventory is S. If inventory level decreases to zero due to service, up to k number of backlogs are allowed to system and the remaining customers are assumed to be lost. Whenever the on-hand inventory level drops to prefixed level s, an order for replenishment is placed. We derived the stability condition and analysed the system using Matrix Analytic Method. Various system performance measures are obtained and numerical illustrations are provided using a suitable cost function.

Keywords -Bernoulli process, Geometric distribution, Lead-time, Matrix Analytic Method, Backlog.

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I. INTRODUCTION

The first reported work on discrete-time queue is Meisling [9]. In this paper, the authors study a single-server queuing system for which time is treated as a discrete variable inter arrival time and service time are geometrically distributed and derived the results in continuous system as the limiting process. Classical inventory theory considers that stock outs generate penalty costs to the firm, often assumed to be proportional to the excess of demand over supply. Due to this the nature of subsequent demand is perturb. This is characterized in [11]. Gross [6] described the development of a continuous-review (s, S) inventory model with complete back ordering and state-dependent, stochastic lead times. The demand on the inventory system is assumed to be a Poisson process, and the order-filling portion of the replenishment lead time is allowed to depend on the number of outstanding orders.

Archibald [3] discussed a continuous review (s, S) policies with lost sales and minimization of minimizes the average stationary cost in an inventory system with constant lead time, fixed order cost, linear holding cost per unit time, linear penalty cost per unit short, discrete compound Poisson demand , lost sales and back ordering. Krishnamoorthy [7] analyzed an (s, S) Inventory system where arrivals of customers form a Poisson process. When inventory level reaches zero due to demands, further demands are sent to a pool which has finite capacity. Deepak [4] considered a queueing system in which work gets postponed due to finiteness of the buffer is considered. When the buffer of finite capacity is full, further arrivals are directed to a pool of customers (postponed work). An arrival encountering then buffer full, will join the pool with some probability else it is lost to the system forever. Manuel [8] considers a continuous review perishable (s, S) inventory system in which the demands arrive according to a Markovian Arrival Process (MAP). The lifetime of items in the stock and the lead time of reorder are assumed to be independently distributed as exponential. Demands that occur during the stock-out periods either enter a pool which has finite capacity or are lost.

Sivakumar [12] considered a continuous review perishable inventory system in which the demands arrive according to a Markovian arrival process MAP. The items in the inventory have shelf life times that are assumed to follow an exponential distribution. The inventory is replenished according to an (s, S) policy and the replenishing times are assumed to follow a phase type distribution. The demands that occur during stock out periods either enter a pool which has finite capacity or leave the system. Any demand that arrives, when the pool is full and the inventory level is zero, is also assumed to be lost. Sivakumar [13] considered a continuous review perishable (s, S) inventory system in which the demands arrive according to a MAP. The life time of each item in the stock and the lead time of orders are assumed to be independently distributed as exponential. They assumed that the demands that occur during stock-out periods either enter a pool of infinite capacity or are lost. The demands in the pool are selected one by one according to FCFS rule when the stock after replenishment is above a prefixed level. In another work by Sivakumar [14] a discrete-time inventory model in which demands arrive according to a discrete Markovian arrival process is analyzed. The inventory is replenished according to an (s, S) policy and the lead time is assumed to follow a discrete phase-type distribution. The demands that occur during stock-out periods either enter a pool which has a finite capacity or leave the system with a predefined probability. A demand that arrives, when the pool is full and the inventory level is zero, is assumed to be lost. This paper is a generalization of [5]. The method adopted in that work is of Neuts [10] and its discrete version is in [1] and in [2].

The rest of the paper is organized as follows. Section II provides mathematical modelling and analysis. Stability condition and steady state probability vector are discussed in section III and IV respectively. System performance measures are described in section V. Cost analysis of the paper is arranged in section VI. Finally, numerical illustrations are included in section VII.

II. MATHEMATICAL MODELLING AND ANALYSIS

The following are the assumptions and notations used in this model.

Assumptions

i. Inter-arrival times of customers are geometrically distributed with parameter p.
ii. Service times are geometrically distributed with parameter q.
iii. Up to k customers are allowed to enter in the system, when the inventory level is zero.
iv. Lead time is geometrically distributed with parameter r.
v. Any demand requires at least one unit of service time (late arrival with delayed access).
vi. Replenishment takes place at the end of slot boundaries.

Notations

$N(n)$: Number ofcustomersinqueueatanepochn.
$I(n)$: Inventory levelattheepochn.
e: $(1,1,1,\dots,1)'$, column vectorof1's ofappropriateorder.

Then $\{(N(n), I(n)); n = 0,1,2,3,..\}$ is a Quasi-Birth Process on the state space $\{(i,j); i \geq 0,0 \leq i \leq S\}$. Now the transition probability matrix of the process is

$$p = \begin{array}{c} 0 \\ 1 \\ \vdots \\ k \\ k+1 \\ k+2 \\ \\ \end{array} \begin{bmatrix} C_1 & C_0 & & & & & \\ B_2 & B_1 & B_0 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & B_2 & B_1 & B_0 & & \\ & & & A_2 & A_1 & A_0 & \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{bmatrix}$$

where the blocks $B_0, A_0, A_{1,i}$and$A_{2,i} (i \geq 0)$ are given by

$$C_1 = \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{bmatrix} \bar{p}\bar{r} & & & & & & pr \\ & \bar{p}\bar{r} & & & & & pr \\ & & \ddots & & & & \vdots \\ & & & \bar{p}\bar{r} & & & pr \\ & & & & \bar{p} & & 0 \\ & & & & & \ddots & \\ & & & & & & \bar{p} \end{bmatrix}$$

$$C_0 = \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{bmatrix} p\bar{r} & & & & & & pr \\ & p\bar{r} & & & & & pr \\ & & \ddots & & & & \vdots \\ & & & p\bar{r} & & & pr \\ & & & & p & & 0 \\ & & & & & \ddots & \\ & & & & & & p \end{bmatrix}$$

$$B_0 = \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{bmatrix} p\bar{r} & & & & & & pr \\ & p\bar{q}\bar{r} & & & & & p\bar{q}r \\ & & \ddots & & & & \vdots \\ & & & p\bar{q}\bar{r} & & & p\bar{q}r \\ & & & & p\bar{q} & & 0 \\ & & & & & \ddots & \\ & & & & & & p\bar{q} \end{bmatrix}$$

$$B_1 = \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{bmatrix} \bar{p}\bar{r} & & & & & & \bar{p}r \\ pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & & & (pq+\bar{p}\bar{q})r \\ & \ddots & \ddots & & & & \vdots \\ & & pq\bar{r} & \bar{p}\bar{q}\bar{r} & & & (pq+\bar{p}\bar{q})r \\ & & & pq & \bar{p}\bar{q} & & 0 \\ & & & & \ddots & \ddots & \vdots \\ & & & & & pq & \bar{p}\bar{q} \end{bmatrix}$$

$$B_2 = \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{bmatrix} 0 & & & & & & 0 \\ \bar{p}q\bar{r} & \ddots & & & & & \bar{p}qr \\ & \ddots & \ddots & & & & \vdots \\ & & \bar{p}q\bar{r} & & & & \bar{p}qr \\ & & & \bar{p}q & \ddots & & 0 \\ & & & & \ddots & \ddots & \\ & & & & & \bar{p}q & 0 \end{bmatrix}$$

$$[A_0]_{1j} = 0, [A_0]_{ij} = [B_0]_{ij} \forall i \neq 1,$$

$$[A_2]_{1j} = 0, [A_2]_{ij} = [B_2]_{ij} \forall i \neq 1 \text{and}$$

$$[A_1]_{11} = \bar{r}, [A_1]_{1S+1} = r, [A_1]_{ij} = [B_1]_{ij} \text{otherwise}$$

III. STABILITY CONDITION

Theorem 3.1 *The above Markov chain is stable, if and only if,* $p < q$
Proof. Consider the matrix $A = A_0 + A_1 + A_2$. Then

$$A = \begin{array}{c} 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{bmatrix} \bar{r} & & & & & & r \\ q\bar{r} & \bar{q}\bar{r} & & & & & r \\ & \ddots & \ddots & & & & \vdots \\ & & q\bar{r} & \bar{q}\bar{r} & & & r \\ & & & q & \bar{q} & & 0 \\ & & & & \ddots & \ddots & \\ & & & & & q & \bar{q} \end{bmatrix}$$

Let π be the steady state probability vector of A. Then the given Markov chain is stable, if and only if $\pi A_0 e < \pi A_2 e$. On simplification we get $p < q$.

IV. Steady-State Analysis

Let $\mathbf{x} = (x_0, x_1, \ldots, x_{k-1}, x_k, \ldots)$ be the steady state probability vector of Q. Under the stability condition, $x_{i'}s$ $(i \geq N)$ are given by

$x_{k+r} = x_k R^r (r \geq 1)$, where R is the minimal non-negative solution of the equation

$$R^2 A_2 + RA_1 + A_0 = R$$

for which the spectral radius is less than 1 and the vectors $x_0, x_1, \ldots, x_k$ are obtained by solving

$$\left.\begin{aligned} &x_0 C_1 + x_1 B_2 = x_0 \\ &x_{J-1} B_0 + x_j B_1 + x_{j+1} B_2 = x_j; (1 \leq j \leq k-1) \\ &x_{k-1} B_0 + x_K (B_1 + RA_2) = x_k \end{aligned}\right\} \quad (1)$$

subject to the normalizing condition

$$\left[\sum_{i=0}^{k-1} x_i + x_k (I - R)^{-1}\right]\mathbf{e} = 1$$

Evaluation of the Rate Matrix R

The rate matrix R is given by $R = \lim_{n\to\infty} R_n$,
where $R_{n+1} = (R_n^2 A_2 + A_0)(I - A_1)^{-1}$ and $R_0 = 0$, The iteration is terminated when $|(R_{n+1} - R_n)| < \varepsilon$.

Computation of Boundary Probabilities

Now the system (1) can be solved using the block Gauss-Seidel iterative method. The vectors $x_0, x_1, \ldots, x_k$ in the$(n+1)$th iteration are given by

$$x_0(n+1) = x_1(n) B_2 (I - C_1)^{-1}$$
$$x_i(n+1) = [x_{i+1}(n) B_2 + x_{i-1}(n+1) B_0](I - B_1)^{-1}; (1 \leq i \leq k-1)$$
$$x_k(n+1) = x_{k-1}(n+1) B_0 (I - B_1 - RA_2)^{-1}$$

All iterations are subject to the normalizing condition (3).

V. System performance measures

We partition the components of $\mathbf{x}$ as $x_i = (y_{i,0}, y_{i,1}, \ldots, y_{i,S})(i \geq 0)$. Then we derive some relevent performance measures of the system under steady state.

i. Expected inventory level, EIL, is given by
ii. $EIL = \sum_{j=1}^{S} \sum_{i=0}^{\infty} y_{i,j}$
iii. Expected number of customers in the system, EC, is given by
$$EC = \sum_{j=1}^{S} \sum_{i=0}^{\infty} i y_{i,j}$$
iv. Expected departure after completing the service, EDS is given by
$$EDS = q \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} y_{i,j}$$
v. The Expected reorder rate, ERO, is given by
$$ERO = q \sum_{i=0}^{\infty} y_{i,s+1}$$
vi. Expected replenishment rate, $ERR = r \sum_{j=0}^{s} \sum_{i=0}^{\infty} y_{i,j}$
vii. Probability that the inventory level zero $= \sum_{i=0}^{\infty} y_{i0}$
viii. Expected loss rate of customers , $ELR = p \sum_{i=k+1}^{\infty} y_{i0}$
ix. Expected number of departure after completing the service, EDS,is given by

$$EDS = q\sum_{i=1}^{\infty}\sum_{j=1}^{S} y_{ij}$$

x. Expected number of customers waiting in the system when the inventory level is zero, $EW0$, is given by

$$EW0 = \sum_{i=1}^{\infty} i y_{i0}$$

VI. Cost Analysis

Define the expected total cost of the system per unit time as

$$ETC = [C_o + \sum_{i=0}^{s} r(S-i)C_p](ERR) + C_{hi}(EIL) + C_{hc}(EW_0) + C_{lc}(ELC)$$

where,

C_o: Setup cost/order
C_p: Procurement cost/unit/unit time
C_{hi}: Inventory holding cost/unit/unit time
C_{hc}: Holding cost of customers/unit/unit time
C_{lc}: Cost due to loss of customers/unit/unit time

VII. Numerical Illustrations

The tables (1) and (2) indicate the relationship between various performance measures with parameters p and q.

p	EIL	EC	EDS	ERO	ABL	EWO	ELR	ETC
0.42	12.034	1.9361	0.44334	0.036437	0.03454	0.035885	0.000455	12.97398
0.44	11.989	2.219	0.46738	0.036332	0.038215	0.040682	0.000621	12.93134
0.46	11.950	2.5421	0.48920	0.036245	0.041426	0.045824	0.000846	12.89555
0.48	11.917	2.9181	0.50868	0.036175	0.043943	0.051566	0.001154	12.86686
0.50	11.890	3.3661	0.52576	0.036118	0.045505	0.058366	0.001576	12.84567
0.52	11.868	3.9138	0.54043	0.036069	0.045849	0.066966	0.002147	12.83162
0.54	11.850	4.6028	0.55271	0.036024	0.044754	0.078506	0.002904	**12.82481**
0.56	11.837	5.4962	0.56266	0.035974	0.042113	0.094674	0.003875	12.82770

Table 1: Effect of p
($q = 0.6$; r = 0.2; $s = 6$; $S = 20$; $k = 6$; $C_o = C_p = C_{hi} = C_{hc} = C_{lc} = 1$)

Table (1) shows that as p increases, expected number of customers, expected number of customers waiting and expected time of re-order increase, but expected inventory level decreases. ETC is optimum(minimum) at $p = 0.54$ and its minimum value is 12.82481.

q	EIL	EC	ERO	EDS	ABL	EWO	ELR	ETC
0.42	9.8002	6.3472	0.038293	0.39308	0.020426	0.053112	0.001475	10.53640
0.44	9.7568	5.1645	0.039875	0.40331	0.024850	0.049042	0.001319	10.51694
0.46	9.7170	4.2942	0.041452	0.41148	0.028968	0.046426	0.001153	10.50243
0.48	9.6806	3.6369	0.043027	0.41761	0.032610	0.044910	0.000995	10.49239
0.50	9.6474	3.1285	0.044600	0.42180	0.035711	0.044197	0.000858	10.48634
0.52	9.6172	2.7265	0.046172	0.42416	0.038285	0.044039	0.000744	**10.48384**
0.54	9.5898	2.4016	0.047743	0.42481	0.040389	0.044236	0.000653	10.48452
0.56	9.5650	2.1337	0.049316	0.42385	0.042091	0.044633	0.000583	10.48804

Table 2: Effect of q

$$(p = 0.4;\ q = 0.2;\ s = 6;\ S = 15;\ C_o = C_p = C_{hi} = C_{hc} = C_{lc} = 1)$$

Table (2) shows that, as q increases, expected inventory level and expected number of customers decrease; whereas the expected re-order rate and expected departure increase.ETC is minimumat $q = 0.52$ and the minimum value is 10.48384.

CONCLUDING REMARKS

In this article, we studied a discrete (s, S) inventory system with a finite backlog. The primary demand constitutes a geometric process. Assumptions that made on the nature of service time and lead-time are geometric distributions. We analyzed the system using Matrix Analytic Method. Stability condition and important system performance measures are derived. The expression for expected total cost and numerical illustrations were also provided. One can extend the present study to another one by considering discrete Phase- type distributions instead of geometric distribution.

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