Fixed Point theorem of Pair of Self Mappings Satisfying Contraction Condition.

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Abstract

In this paper we prove a common fixed point Result of a pair of self mappings satisfying contractive condition in Complete Metric Space.

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1 Introduction

The Fixed point Theory has many applications in various Branches of Engineering and Science field. S.Banach [1]derived a well known theorem for a contraction mapping in a Complete metric space.Which states that,"A contraction has a Unique fixed point theorem in a Complete metric space." After that many authors proved fixed point theorems for mappings satisfying certain contraction conditions.In 1968 R.Kanan [5] introduced another type of map called as Kanan Map and investigated Unique fixed point theorem in complete metric space.In 1973 B.K. Das and Sattya Gupta [3] Generalized Banach contraction principle in terms of rational expression .Recently V.V. Latpate and Dolhare U.P.[4] generalized fixed point theorem in Generalized metric space.

In this paper we have proved a common fixed point theorem for a pair of self maps which is based on B.k. Das and Satya Gupta [3] .

2 Some Preliminaries

2.1 Definition

Let X be a non empty set .A mapping $d : X \times X \to R$ is said to be a metric or a distance function if it satisfies following conditions .

1. $d(x, y)$ *is non negative.*
2. $d(x, y) = 0$ if *and only* $if x$ *and* y *coincides i.e.* $x = y$
3. $d(x, y) = d(y, x)$ *(symmetry)*
4. $d(x, y) \leq d(x, z) + d(z, y)$ *(triangle inequality)*

Then the function d is referred to as metric on X*.And* (X, d) *or simply* X *is said to as metric space.*

2.2 Definition

A metric space (x, d) *is said to be a complete metric space if every cauchy sequence in* X *converges to a point of* X*.*

2.3 Definition

If (X, d) *be a complete metric space and a function* $F : X \to X$ *is said to be a contraction map if*
$d(F(x), F(y)) \leq \beta\ d(x, y)$ *for all* $x, y\epsilon\ X$ *and for* $0 < \beta < 1$

2.4 Definition

Let $F : X \to X$ *, then* $x\epsilon X$ *is said to be a fixed point of* F *if* $F(x) = x$*.*
Ex 2.1 If $f(x) = sin(x)$, then o is the fixed point of f since f(0)=0

2.5 Definition

Let X *be a metric space and if* F_1 *and* F_2 *be any two maps .An element* $a\epsilon X$ *is said to be a common fixed point of* F_1 *and*F_2 *if* $F_1(a) = F_2(a)$
Ex 2.2 If $F_1(x) = Sin(x)$ and $F_2(x) = Tan(x)$. Then o is the common fixed point of F_1 *and*F_2 since $F_1(0) = Sin(0) = 0$ and $F_2(0) = Tan(0) = 0$.

3 Main Result

Theorem 3.1 *Let* (X, d) *be a complete metric space . A mapping* $F : X \to X$ *be continuous and* F *satisfies the condition*

$$d(F(x), F(y) \leq \frac{\mu\ d(x, F(x))\ d(y, F(y))}{d(x, y)} + \mu\ \lambda d(x, y) + \mu d(x, y) \qquad (1)$$

for all $x, y \epsilon X$, $x \neq y$ *and for* μ , λ $\epsilon[0, \frac{1}{2})$ *with* $\mu + \lambda < \frac{1}{2}$ *and* $\mu\ \lambda < \frac{1}{2}$. *Then* F *has a Unique fixed point in X.*

Proof:- Let us consider $x_0 \epsilon X$ an arbitrary point and Let x_n be a sequence in X s.t. $f^n(x_0) = x_n$ for n be a positive integer.

If $x_n = x_{n+1}$ for some n then the result is obvious.Let $x_n \neq x_{n+1}$ for all n.

consider $d(x_{n+1}, x_n) = d(F(x_n), F(x_{n-1}))$

$$d(x_{n+1}, x_n) \leq \frac{\mu\ d(x_n, F(x_n))\ d(x_{n-1}, F(x_{n-1}))}{d(x_n, x_{n-1})} + \mu\ \lambda d(x_n, x_{n-1}) + \mu d(x_n, x_{n-1})$$

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \left(\frac{\mu\lambda}{1-\mu}\right) d(x_n, x_{n-1}) \\ &\leq \left(\frac{\mu\lambda}{1-\mu}\right)^2 d(x_{n-1}, x_{n-2}) \\ &\leq . \\ &\leq . \\ &\leq . \\ &\leq \left(\frac{\mu\lambda}{1-\mu}\right)^n d(x_1, x_0) \end{aligned}$$

for $m \geq n$,using the triangle inequality , we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq (\alpha^n + \alpha^n + 1 + ... + \alpha^m - 1)\ d(x_1, x_0)\ where\ alpha = \frac{\mu\lambda}{1-\mu} \\ &\leq \frac{\alpha^n}{1-\alpha}\ d(x_1, x_0) \end{aligned}$$

But $\frac{\alpha^n}{1-\alpha} \to 0\ as\ m, n \to \infty$

$R.H.S. \to 0$

x_n be a Cauchy sequence in X and Since X is complete . converges to some element of X.

say $p \epsilon X$ $x_n \to p$ and since F is continuous .

$$\begin{aligned} consider\ F(p) &= F\big(\lim_{n\to\infty} x_n\big) \\ &= \lim_{n\to\infty} F(x_n) \\ &= p \end{aligned}$$

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p is a fixed point of F in X.
For Uniqueness if q be distinct point F from p in X.
then F(q)=q.
consider

$$\begin{aligned} d(q,p) &= d(F(q),F(p)) \\ &\leq d(F(q),F(p)) \leq \frac{\mu\ d(q,F(q))\ d(p,F(p))}{d(q,p)} + \mu\ \lambda d(q,p) \\ &= \mu\ \lambda d(q,p) + \mu d(q,p) \end{aligned}$$

$< d(q,p)$ which is a contradiction.
p is a fixed point of F in X.
Theorem 3.2 *Let $F : X \to X$ where X be a complete metric space such that F satisfies (1) and if for some $n\epsilon I^+$ if F^n is uniformly continuous ,then F has a unique fixed point.*

Now we prove a unique common fixed theorem of a pair of self mappings defined on a complete metric space (X,d) which satisfies (1).

Theorem 3.3 Let F_1,F_2 be two mappings which maps $X \times X$ to R ,where (X,d) be a complete metric space satisfying the condition

1. $d(F(x),F(y) \leq \frac{\mu\ d(x,F(x))\ d(y,F(y))}{d(x,y)} + \mu\ \lambda d(x,y) + \mu d(x,y)$ for all $x, y\epsilon X$, $x \neq y$ and for μ , $\lambda\ \epsilon[0,\frac{1}{2})$ with $\mu\ + \lambda < \frac{1}{2}$ and $\mu\ \lambda < \frac{1}{2}$.

2. there is an $x_0\epsilon X$, the sequence x_n is such that $x_n = F_1(x_{n-1})$, if n is even
and $x_n = F_2(x_{n-1})$, if n is odd, and $x_n \neq x_{n+1}$ for all n.

3. F_1F_2 is continuous on X.Then $F_1\,F_2$ has a unique common fixed point in X.
Proof consider

$$\begin{aligned} d(x_{2n+1},x_{2n}) &= d(F_2(x_{2n}),F_1(x_{2n-1})) \\ &\leq \frac{\mu\ d(x_{2n},F_2(x_{2n}))\ d(x_{2n-1},F_1(x_{2n-1}))}{d(x_{2n},x_{2n-1})} + \mu\ \lambda d(x_{2n},x_{2n-1}) \\ &+ \mu d(x_{2n},x_{2n-1}) \end{aligned}$$

which gives

$$
\begin{aligned}
d(x_{2n+1}, x_{2n}) &\leq \left(\frac{\mu\lambda}{1-\mu}\right) d(x_{2n}, x_{2n-1}) \\
&\leq . \\
&\leq . \\
&\leq . \\
&\leq \left(\frac{\mu\lambda}{1-\mu}\right)^{2n} d(x_0, x_1) \\
d(x_{2n+2}, x_{2n+1}) &\leq \left(\frac{\mu\lambda}{1-\mu}\right)^{2n+1} d(x_0, x_1)
\end{aligned}
$$

For $m \geq n$,using triangle inequality, we have
Consider

$$
\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\
&\leq (\alpha^n + \alpha^{n+1} + ... + \alpha^{m-1})\ d(x_1, x_0)\ where\ alpha = \frac{\mu\lambda}{1-\mu} \\
&\leq \frac{\alpha^n}{1-\alpha}\ d(x_1, x_0)
\end{aligned}
$$

$\frac{\alpha^n}{1-\alpha} \to 0\ as\ m, n \to 0$ Since X is complete,there exists p in X such that $x_n \to p$. $\therefore$ its subsequence x_{n_k} also converges to p where $n_k = 2k$
Since F_1F_2 be continuous on X. $\therefore$ We have

$$
\begin{aligned}
F_1F_2(p) &= F_1F_2(\lim_{k\to\infty} x_{n_k}) \\
&= \lim_{k\to\infty} (x_{n_{k+1}}) \\
&= p
\end{aligned}
$$

If possible suppose that $F_2(p) \neq p$, then

$$
\begin{aligned}
d(F_2(p), p) &= d(F_2(u), F_1F_2(u)) \\
&\leq \frac{\mu d(p, F_2(p)).d(F_2(p), F_1F_2(p))}{d(p, F_2(p))} + \mu\lambda d(p, F_2(p)) + d(p, F_2(p)) \\
&\leq (\mu + \lambda) d(p, F_2(p))
\end{aligned}
$$

which is impossible,since $\mu + \lambda \leq 1$
$\therefore$ we have $F_2(p) = p$. Similarly,

$$
\begin{aligned}
d(F_1(p), p) &= d(F_1(F_2(p)), p) \\
&= d(p, p) \\
&= 0
\end{aligned}
$$

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as $d(F_1(p), p) = 0 \therefore F_1(p) = p \therefore$ p is a common fixed point of F_1 *and* F_2.
For Uniqueness if possible suppose that $q \epsilon X$ is another fixed point of F_1 which is distinct from p.
Consider

$$\begin{aligned}
d(q,p) &= d(F_1(q), F_2(p)) \\
&\leq \frac{\mu\ d(q, F_1(q))\ d(p, F_2(p))}{d(q,p)} + \mu\ \lambda d(q,p) \\
d(q,p) &\leq 0 + \mu\ \lambda d(q,p) < d(q,p) \\
\therefore d(q,p) &< d(q,p)
\end{aligned}$$

Which is not possible.
$\therefore$ p is a unique fixed point of F_1.
Similarly we can prove that p is also a unique fixed point of F_2. $\therefore$ p is the unique common fixed point of F_1 and F_2

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