Generalization of Hameed Obiedat Common Fixed Point Result in G-Metric Space.

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Abstract

In this paper we Generalize Hameed Obiedat fixed point results of pair of self maps satisfying involving maximum and minimum function in G-Metric space .

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1 Introduction

Z. Mustafa and B.Sims [1] introduced a generalized Metric Space namely G-Metric space.For more detail information one can refer [2, 3, 4, 5].Ray (1976) [6] proved common fixed point theorem for the pair of self maps on complete Metric space X into itself.In 1977 B.Fisher [7] proved some results of common fixed points for pair of continuous maps which satisfies a contractive condition in terms of Maximum function.Then Z.Mustafa et. and all [8] proved some common fixed point theorems of pair of mapping satisfy contractive condition in terms of maximum function.

2 Preliminaries

Definition 2.1. *Let X be a non empty set and $G : X^3 \to R^+$ which satisfies the following conditions*

1. $G(a, b, c) = 0$ $if a = b = c$ *i.e. every* a, b, c *in* X *coincides.*

2. $G(a, a, b) > 0$ for *every* a, b, c ϵX *s.t.* $a \neq b$

3. $G(a, a, b) \leq G(a, b, c)$, $\forall a, b, c$ ϵ X *s.t.* $c \neq b$

4. $G(a, b, c) = G(b, a, c) = G(c, b, a) = ...$
   (symmetrical in all three variables)

5. $G(a, b, c) \leq G(a, x, x) + G(x, b, c)$, *for all a,b,c,x in X*
   (rectangle inequality)

   Then the function G is said to be generalized metric or simply G-metric on X and the pair (X,G) is said to be G-metric space.

Example 2.2. *Let* $G : X^3 \to R^+$ *s.t.* $G(a, b, c) =$ *perimeter of the triangle with vertices at a,b,c in* R^2 *, also by taking p in the interior of the triangle then rectangle inequality is satisfied and the function G is a G-metric on X.*

Remark 2.3. *G-metric space is the generalization of the ordinary metric space that is every G-metric space is (X,G) defines ordinary metric space (X,* d_G *) by* d_G *(a,b)=*$G(a, b, b)+ G(a, a, b)$

Example 2.4. *Let (X,d) be the usual metric space . Then the function* $G : X^3 \to R^+$ *defined by*

$$G(a, b, c) = max.\{d(a, b), d(b, c), d(c, a)\}$$

for all a,b,c ϵ *X is a G-metric space.*

Definition 2.5. *Let (X,G) be a G-metric space, Let* $\{ a_n \}$ *be a sequence of elements in X .The sequence* $\{ a_n \}$ *is said to be G-convergent to a if*

$$lim_{m,n\to\infty} G(a, a_n, a_m) = 0$$

i.e for every $\epsilon > 0$ *there is N s.t.* $G(a, a_n, a_m) < \epsilon$ *for all m,n* $\geq N$ *It is denoted as* $a_n \to a$ *or* $lim_{n\to\infty} a_n = a$

Definition 2.6. *Let (X,G) be a G-metric space a sequence* $\{a_n\}$ *is called G-Cauchy if , for each* $\epsilon > 0$ *there is an* $N \; \epsilon I^+$ *(set of positive integers) s.t.*

$$G(a_n, a_m, a_l) < \epsilon \; for \; all \; n, m, l \geq N$$

Theorem 2.7. *If* P *and* Q *are two continuous mappings of the complete metric space* X *into itself s.t.* $\rho(P^2x, Q^2y)) \leq cmax.\{\rho(Px, Qy), \rho(x, y)\}$ *for all* $x, y \in X$ *, where* $0 \leq c < 1$ *.Then* P *and* Q *have a unique fixed point* z *.*

Theorem 2.8. *If* (X, G) *be a complete G-Metric space and* F *be a self mapping of* X *which satisfies the following condition for all* $a, b, c \in X$ $G(Fa, Fb, Fc) \leq \alpha G(a, b, c)$ *for* $0 \leq 1$ *.Then F has a unique fixed point.*

Theorem 2.9. *Let* (X, G) *be a complete G-metric space, and F be a self map which satisfy the condition*

$$G(F(a), F(b), F(c)) \leq \alpha max \begin{Bmatrix} G(a, F(a), F(a)) \\ G(b, F(b), F(b)) \\ G(c, F(c), F(c)) \end{Bmatrix}$$

or

$$G(F(a), F(b), F(c)) \leq \alpha max \begin{Bmatrix} G(a, a, F(a)) \\ G(b, b, F(b)) \\ G(c, c, F(c)) \end{Bmatrix}$$

for all $a, b, cin \; X$ *,for* $\alpha \in [0, 1)$ *.Then F has a unique fixed point.*

Hameed Obiedat [9] Proved following Theorem.

Theorem 2.10. *Let* (X, G) *be a complete G-metric space, and Let* $F : X \rightarrow X$ *be a map which satisfies* $G(F(a), F(b), F(c)) \leq \beta G(a, b, c) + \alpha \; max. \begin{Bmatrix} G(a, F(a), F(a)), G(b, F(b), F(b)) \\ G(c, F(c), F(c)) \end{Bmatrix}$ *for all* $a, b, c \in X$ *,where* $0 \leq \beta + \alpha < 1$ *.Then* F *has a unique fixed point (say v), and F is G-continuous at u.*

3 Main Result

Now we prove result of common fixed point theorem for pair of self maps for following contraction.

Theorem 3.1. *If (X, G) be a complete G-Metric space.If the pair of self maps P, Q satisfy*

$$\begin{aligned}
max \begin{Bmatrix} G(P(a), Q(b), Q(b)), \\ G(Q(a), P(b), P(b)) \end{Bmatrix} &\leq \alpha_1 \; G(a, b, b) \\
&+ \alpha_2 \; min \begin{Bmatrix} G(a, Q(b), Q(b)) + G(b, P(a), P(a))) \\ G(a, P(b), P(b)) + G(b, Q(a), Q(a)) \end{Bmatrix} \\
&+ \alpha_3 \; min \begin{Bmatrix} G(a, a, P(a)) + G(b, Q(b), Q(b)), \\ G(a, Q(a), Q(a)) + G(b, P(b), P(b)) \end{Bmatrix} \quad (3.1) \\
&+ \alpha_4 \; G(a, b, b) \quad (3.2)
\end{aligned}$$

for all a, b in X , where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0$ s.t. $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 < 1$. Then P and Q have a unique common fixed point in X.

Proof. Let a_0 be an arbitrary element of X.We define a sequence $\{a_n\}$ by

$$a_n = \begin{cases} P(a_{n-1}), & if \; n \; is \; odd \\ Q(a_{n-1}), & if \; n \; is \; even \end{cases}$$

If $n \in I^+$ be an odd positive integer then by using (3.1), we have

$$\begin{aligned}
G(a_n, a_{n+1}, a_{n+1}) &= G(P(a_{n-1}, Q(a_n), Q(a_n))) \\
&\leq max. \begin{Bmatrix} G(P(a_{n-1}), Q(a_n), Q(a_n)) \\ G(Q(a_{n-1}), P(a_n), P(a_n)) \end{Bmatrix} \\
&\leq \alpha_1 G(a_{n-1}, a_n, a_n) \\
&+ \alpha_2 \, min \begin{Bmatrix} G(a_{n-1}, Q(a_n), Q(a_n)) + G(a_n, P(a_{n-1}), P(a_{n-1})) \\ G(a_{n-1}, P(a_n), P(a_n)) + G(a_n, Q(a_{n-1}), Q(a_{n-1})) \end{Bmatrix} \\
&+ \alpha_3 \, min \begin{Bmatrix} G(a_{n-1}, P(a_{n-1}), P(a_{n-1})) + G(a_n, Q(a_n), Q(a_n)) \\ G(a_{n-1}, Q(a_{n-1}), Q(a_{n-1})) + G(a_n, P(a_n), P(a_n)) \end{Bmatrix} \\
&+ \alpha_4 G(a_{n-1}, a_n, a_n)
\end{aligned}$$

Thus,

$$\begin{aligned}
G(a_n, a_{n+1}, a_{n+1}) &\leq \alpha_1 \; G(a_{n-1}, a_n, a_n) \\
&+ \alpha_2 \; \{G(a_{n-1}, Q(a_n), Q(a_n)) + G(a_n, P(a_{n-1}), Q(a_{n-1}))\} \\
&+ \alpha_3 \; \{G(a_{n-1}, P(a_{n-1}), P(a_{n-1})) + G(a_n, Q(a_n), Q(a_n))\} \\
&= \alpha_1 G(a_{n-1}, a_n, a_n) \\
&+ \alpha_2 \; \{G(a_{n-1}, a_{n+1}, a_{n+1}) + G(a_n, a_n, a_n)\} \\
&+ \alpha_3 \; \{G(a_{n-1}, a_n, a_n) + G(a_n, a_{n+1}, a_{n+1})\} \\
&\leq \alpha_1 \; G(a_{n-1}, a_n, a_n) \\
&+ \alpha_2 \; \{G(a_{n-1}, a_n, a_n) + G(a_n, a_{n+1}, a_{n+1})\} \\
&+ \alpha_3 \; \{G(a_{n-1}, a_n, a_n) + G(a_n, a_{n+1}, a_{n+1})\} \\
&+ \alpha_4 \; G(a_{n-1}, a_n, a_n)
\end{aligned}$$

This gives, $G(a_n, a_{n+1}, a_{n+1}) \leq \frac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_2-\alpha_3} G(a_{n-1}, a_n, a_n)$

If $n \in N$ is an even then by using (3.1), we get

$$
\begin{aligned}
G(a_n, a_{n+1}, a_{n+1}) &= G(Q(a_{n-1}), P(a_n), P(a_n)) \\
&\leq max \begin{Bmatrix} G(P(a_{n-1}), Q(a_n), Q(a_n)) \\ G(Q(a_{n-1}), P(a_n), P(a_n)) \end{Bmatrix} \\
&\leq \alpha_1 \ G(a_{n-1}, a_n, a_n) \\
&+ \alpha_2 \ min \ \begin{Bmatrix} G(a_{n-1}, Q(a_n), Q(a_n)) + G(a_n, P(a_{n-1}), P(a_{n-1})) \\ G(a_{n-1}, P(a_n), P(a_n)) + G(a_n, Q(a_{n-1}), Q(a_{n-1})) \end{Bmatrix} \\
&+ \alpha_3 \ min \begin{Bmatrix} G(a_{n-1}, P(a_{n-1}), P(a_{n-1})) + G(a_n, Q(a_n), Q(a_n)) \\ G(a_{n-1}, Q(a_{n-1}), P(a_{n-1})) + G(a_n, P(a_n), P(a_n)) \end{Bmatrix} \\
&+ \alpha_4 \ G(a_{n-1}, a_n, a_n) \\
&\leq \alpha_1 \ G(a_{n-1}, a_n, a_n) \\
&+ \alpha_2 \ \{G(a_{n-1}, P(a_n), P(a_n)) + G(a_n, Q(a_{n-1}), Q(a_{n-1}))\} \\
&+ \alpha_3 \ \{G(a_{n-1}, Q(a_{n-1}), Q(a_{n-1})) + G(a_n, P(a_n), P(a_n))\} \\
&+ \alpha_4 \ G(a_{n-1}, a_n, a_n)
\end{aligned}
$$

$$
\begin{aligned}
G(a_n, a_{n+1}, a_{n+1}) &\leq \alpha_1 \ G(a_{n-1}, a_n, a_n) \\
&+ \alpha_2 \ \{G(a_{n-1}, a_{n+1}, a_{n+1}) + G(a_n, a_n, a_n)\} \\
&+ \alpha_3 \ \{G(a_{n-1}, a_n, a_n) + G(a_n, a_{n+1}, a_{n+1})\} \\
&+ \alpha_4 \ G(a_{n-1}, a_n, a_n) \\
&\leq \alpha_1 \ G(a_{n-1}, a_n, a_n) \\
&+ \alpha_2 \ \{G(a_{n-1}, a_n, a_n) + G(a_n, a_{n+1}, a_{n+1})\} \\
&+ \alpha_3 \ \{G(a_{n-1}, a_n, a_n) + G(a_n, a_{n+1}, a_{n+1})\} \\
&+ \alpha_4 \ G(a_{n-1}, a_n, a_n)
\end{aligned}
$$

which gives that, $G(a_n, a_{n+1}, a_{n+1}) \leq \frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{1-\alpha_2-\alpha_3-\alpha_4} G(a_{n-1}, a_n, a_n)$ for any positive integer n , We have

(3.3) $$G(a_n, a_{n+1}, a_{n+1}) \leq \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_3 - \alpha_4} G(a_{n-1}, a_n, a_n)$$

Let $\beta = \frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{1-\alpha_2-\alpha_3-\alpha_4}$, $\therefore 0 \leq \beta < 1$ as $\alpha_1, \alpha_2, \alpha_3 \geq 0$ and $\alpha_1 + 2\alpha_2 + 2\alpha_3 < 1$ with this (3.2) becomes

$$G(a_n, a_{n+1}, a_{n+1}) \leq \beta G(a_{n-1}, a_n, a_n) \tag{3.4}$$

by using relation (3.3) repeatedly, we get

$$G(a_n, a_{n+1}, a_{n+1}) \leq \beta^n G(a_0, a_1, a_1) \tag{3.5}$$

By using rectangular inequality repeatedly and using (3.4), we get, for all $n, m \in N$ s.t. $m > n$

$$\begin{aligned} G(a_n, a_m, a_m) &\leq G(a_n, a_{n+1}, a_{n+1}) + G(a_{n+1}, a_{n+2}, a_{n+2}) \\ &+ G(a_{n+2}, a_{n+3}, a_{n+3}) + ... + G(a_{m-1}, a_m, a_m) \\ &\leq (\beta^n + \beta^{n+1} + ... + \beta^{m-1}) G(a_0, a_1, a_1) \\ &\leq \frac{\beta^n}{1-\beta} G(a_0, a_1, a_1) \end{aligned}$$

taking limit as $n, m \to \infty$, we get $lim G(a_n, a_m, a_m) = 0$, since $lim_{n\to\infty} \frac{\beta^n}{1-\beta} = 0$ $\therefore$ limit of R.H.S. is 0. $\therefore \{a_n\}$ is a G-Cauchy sequence, since X is G-complete. $\therefore$ there exists some $v \in X$ s.t. $\{a_n\}$ converges to $v \in X$. Again by using rectangle inequality and by equation (3.1), we get

$$
\begin{aligned}
G(v,Q(b),Q(b)) &\le G(v,a_{2n+1},a_{2n+1})+G(a_{2n+1},Q(v),Q(v))\\
&= G(v,a_{2n+1},a_{2n+1})+G(P(a_2n),Q(v),Q(v))\\
&\le G(v,a_{2n+1},a_{2n+1})+max\begin{Bmatrix}G(P(a_{2n}),Q(v),Q(v)),\\ G(Q(a_{2n}),P(v),P(v))\end{Bmatrix}\\
&\le G(v,a_{2n+1},a_{2n+1})+\alpha_1\ G(a_{2n},v,v)\\
&+ \alpha_2\ min\begin{Bmatrix}G(a_{2n},Q(v),Q(v))+G(v,P(a_{2n}),P(a_{2n})),\\ G(a_{2n},P(v),P(v))+G(v,Q(a_{2n}),Q(a_{2n}))\end{Bmatrix}\\
&+ \alpha_3\ min\begin{Bmatrix}G(a_{2n},P(a_{2n}),P(a_{2n}))+G(v,Q(v),Q(v)),\\ G(a_{2n},Q(a_{2n}),Q(a_{2n}))+G(v,P(v),P(v))\end{Bmatrix}\\
&+ \alpha_4\ G(a_{2n},v,v)\\
&\le G(v,a_{2n+1},a_{2n+1})+\alpha_1\ G(a_{2n},v,v)\\
&+ \alpha_2\ G(a_{2n},Q(v),Q(v))+G(v,P(a_{2n}),P(a_{2n}))\\
&+ \alpha_3\ G(a_2n,P(a_{2n}),P(a_{2n}))+G(v,Q(v),Q(v))\\
&+ \alpha_4\ G(a_{2n},v,v)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
G(v,Q(v),Q(v)) &\le G(v,a_{2n+1},a_{2n+1})+\alpha_1\ G(a_{2n},v,v)\\
&+ \alpha_2\ G(a_{2n},Q(v),Q(v))+G(v,a_{2n+1},a_{2n+1})\\
&+ \alpha_3\ G(a_{2n},a_{2n+1},a_{2n+1})+G(v,Q(v),Q(v))\\
&+ \alpha_4\ G(a_{2n},v,v)
\end{aligned}
$$

taking limit as $n\to\infty$, and given that the function G is continuous on its variables , $\therefore$ we have $G(v,Q(v),Q(v))\le(\alpha_1+\alpha_2)G(v,Q(v),Q(v))$ But, $0\le(\alpha_1+\alpha_2)<1$, $\therefore G(v,Q(v),Q(v))\le G(v,Q(v),Q(v))$, which is impossible. $\therefore G(v,Q(v),Q(v))=0$ $\therefore$ This gives $Q(v)=v$.In the same way, we can prove that $P(v)=v$.$\therefore$ u is a common fixed point of P and Q. To prove uniqueness, if possible suppose that there exists some

$u \in X$ s.t. P(u)=Q(u)=u. Then

$$
\begin{aligned}
G(v,u,u) &= G(P(v),Q(u),Q(u)) \\
&\leq max \begin{Bmatrix} G(P(v),Q(u),Q(u)), \\ G(Q(v),P(u),P(u)) \end{Bmatrix} \\
&\leq \alpha_1 G(v,u,u) \\
&+ \alpha_2 \; min \begin{Bmatrix} G(v,Q(u),Q(u)) + G(u,P(v),P(v)), \\ G(v,P(u),P(u)) + G(u,Q(v),Q(v)) \end{Bmatrix} \\
&\leq \alpha_3 \; min \begin{Bmatrix} G(v,P(v),P(v)) + G(u,Q(u),Q(u)), \\ G(v,Q(v),Q(v)) + G(u,P(u),P(u)) \end{Bmatrix} \\
&+ \alpha_4 G(v,u,u) \\
&= \alpha_1 G(v,u,u) \\
&+ \alpha_2 G(v,u,u) + G(u,v,v) \\
&+ \alpha_3 G(v,v,v) + G(u,u,u) \\
&+ \alpha_4 G(v,u,u)
\end{aligned}
$$

Which gives,

$$G(v,u,u) \leq \frac{\alpha_2}{1-\alpha_1-\alpha_2} G(u,v,v)$$

Similarly, we get

$$G(u,v,v) \leq \frac{\alpha_2}{1-\alpha_1-\alpha_2} G(v,u,u)$$

$\therefore$ from above two inequalities,we have

$$G(v,u,u) \leq (\frac{\alpha_2}{1-\alpha_1-\alpha_2})^2 G(v,u,u)$$

Since $0 \leq \frac{\alpha_2}{1-\alpha_1-\alpha_2} < 1$ $\therefore$ this gives $u = v$. □

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