

STUDY OF COMPATIBLE MAPS AND FIXED POINT THEOREM IN G-METRIC SPACE.

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Abstract

In this paper we obtain generalized Common fixed point theorem of Compatible Maps in G-Metric Space .

Keywords: G-Metric Space, G-convergent Sequence, G-Cauchy sequence, Compatible Mappings .

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1 Introduction

In 2006 Mustafa Z. and Sims B. [1] invented the concept of G-Metric space. In 1986 G.Jungck [2] introduced the concept of compatibility and proved some common fixed point theorems. In 2012 Manoj Kumar [3] discussed the concept of compatible maps in G-Metric space and proved some theorems of common fixed points of compatible maps. Latpate V.V. and Dolhare U.P. [4] proved one result of common fixed point theorem for compatible mappings in G-Metric space. Jungck's [5] common fixed point theorem has been generalized and modified by several authors [7, 8, 2, 9, 10].

2 Preliminaries

Definition 2.1. Let X be a non empty set and $G : X^3 \rightarrow R^+$ which satisfies the following axioms

1. $G(a, b, c) = 0$ if $a = b = c$ i.e. every a, b, c in X coincides.
2. $G(a, a, b) > 0$ for every $a, b, c \in X$ s.t. $a \neq b$
3. $G(a, a, b) \leq G(a, b, c)$, $\forall a, b, c \in X$ s.t. $c \neq b$
4. $G(a, b, c) = G(b, a, c) = G(c, b, a) = \dots\dots\dots$
5. $G(a, b, c) \leq G(a, x, x) + G(x, b, c)$, for all a, b, c, x in X
(rectangle inequality)

Then the function G is said to be G -metric on X and the pair (X, G) is said to be G -metric space.

Example 2.2. Let $G : X^3 \rightarrow R^+$ s.t. $G(a, b, c) =$ perimeter of the triangle with vertices at a, b, c in R^2 , also by taking p in the interior of the triangle then rectangle inequality is satisfied and the function G is a G -metric on X .

Definition 2.3. Let (X, G) be a G -metric space, Let $\{a_n\}$ be a sequence of elements in X . The sequence $\{a_n\}$ is said to be G -convergent to a if

$$\lim_{m, n \rightarrow \infty} G(a, a_n, a_m) = 0$$

i.e for every $\epsilon > 0$ there is N s.t. $G(a, a_n, a_m) < \epsilon$ for all $m, n \geq N$ It is denoted as $a_n \rightarrow a$ or $\lim_{n \rightarrow \infty} a_n = a$

Definition 2.4. Let (X, G) be a G -metric space a sequence $\{a_n\}$ is called G -Cauchy if, for each $\epsilon > 0$ there is an $N \in I^+$ (set of positive integers) s.t.

$$G(a_n, a_m, a_l) < \epsilon \text{ for all } n, m, l \geq N$$

Definition 2.5. [3] Let S and T be two self mappings on a G -metric space (X, G) . Then mappings S and T are said to be compatible if $\lim_{n \rightarrow \infty} G(STx_n, STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X s.t. $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = z$ for some $z \in X$ for some z in X .

Example 2.6. If $X = [-1, 1]$ and G be a G -metric space s.t. $G : X^3 \rightarrow R^+$ defined by

$$G(x_1, y_1, z_1) = (|x_1 - y_1| + |y_1 - z_1| + |z_1 - x_1|)$$

for all $x_1, y_1, z_1 \in X$. Then X is a G -Metric space. We define $S(x_1) = \frac{x_1}{6}$ and $T(x_1) = \frac{x_1}{2}$. If S is Continuous and $S(X) \subseteq T(X)$.

Here $G(Sx_1, Sy_1, Sz_1) \leq \beta G(Tx_1, Ty_1, Tz_1)$ is true for all $x_1, y_1, z_1 \in X$, $\frac{1}{3} \leq \beta < 1$ and 0 is the common fixed point of S and T which is Unique.

In 2017, Latpate V.V. and Dolhare U.P [4] proved common fixed point theorem for pair of compatible maps in G -Metric space.

Theorem 2.7. Let X be a complete G -metric space. $S, T : X \rightarrow X$ be two compatible maps on X and which satisfies the following conditions,

(i) $S(X) \subseteq T(X)$,

(ii) S or T is G -continuous,

(iii) $G(Sa, Sb, Sc) \leq \alpha G(Sa, Tb, Tc) + \beta G(Ta, Sb, Tc) + \gamma G(Ta, Tb, Sc) + \delta G(Sa, Tb, Tc)$

for every a, b, c in X and $\alpha, \beta, \gamma, \delta \geq 0$ with $0 \leq \alpha + 3\beta + 3\gamma + 3\delta < 1$. Then S and T have unique common fixed point in X .

Now, we generalize our Main result, for the compatible maps.

3 Main Result

Theorem 3.1. Let X be a complete G -Metric space, Let T and S be self mappings of X satisfying the following conditions,

1. $T(X) \subseteq S(X)$

2. T or S is continuous,

$$3. \quad G(Ta, Tb, Tc) \leq k \max \left\{ \begin{array}{l} G(Sa, Sb, Sc), G(Sa, Ta, Ta), G(Sa, Tb, Tb) \\ G(Sa, Tc, Tc), G(Sb, Tb, Tb), G(Sb, Ta, Ta) \\ G(Sb, Tc, Tc), G(Sc, Tc, Tc), G(Sc, Ta, Ta) \\ G(Sc, Tb, Tb) \end{array} \right\}$$

for all $a, b, c \in X$, where $0 \leq k < \frac{1}{4}$. Then T and S have unique common fixed point in X . Provided T and S are compatible maps.

Proof. Let a_0 be an arbitrary point in X . By using equation (1), one can choose a point a_1 in X s.t. $Ta_0 = Sa_1$. In general we can choose a point a_{n+1} s.t. $b_n = Ta_n = Sa_{n+1}$, $n = 0, 1, 2, 3, \dots$ from (3), we have

(3.1)

$$G(Ta_n, Ta_{n+1}, Ta_{n+1}) \leq k \max \left\{ \begin{array}{l} G(Sa_n, Sa_{n+1}, Sa_{n+1}), G(Sa_n, Ta_n, Ta_n), G(Sa_n, Ta_{n+1}, Ta_{n+1}), \\ G(Sa_n, Ta_{n+1}, Ta_{n+1}), G(Sa_{n+1}, Ta_{n+1}, Ta_{n+1}), G(Sa_{n+1}, Ta_n, Sa_n), \\ G(Sa_{n+1}, Ta_{n+1}, Ta_{n+1}), G(Sa_{n+1}, Ta_{n+1}, Ta_{n+1}), G(Sa_{n+1}, Ta_n, Ta_n), \\ G(Sa_{n+1}, Ta_{n+1}, Ta_{n+1}) \end{array} \right\}$$

or

$$G(b_n, b_{n+1}, b_{n+1}) \leq k \max \{G(b_{n-1}, b_n, b_n), G(b_{n-1}, b_{n+1}, b_{n+1}), G(b_n, b_{n+1}, b_{n+1})\}$$

Possibility 1 If

$$\max \{G(b_{n-1}, b_n, b_n), G(b_{n-1}, b_{n+1}, b_{n+1}), G(b_n, b_{n+1}, b_{n+1})\} = G(b_{n-1}, b_n, b_n)$$

then using (3), we get

$$G(b_n, b_{n+1}, b_{n+1}) \leq kG(b_{n-1}, b_n, b_n)$$

, continuing in the same way, we have

$$G(b_n, b_{n+1}, b_{n+1}) \leq k^n G(b_0, b_1, b_1)$$

.Therefore for all $n, m \in N$ $n < m$ and by using rectangle inequality, we get

$$\begin{aligned} G(b_n, b_m, b_m) &\leq G(b_n, b_{n+1}, b_{n+1}) + G(b_{n+1}, b_{n+2}, b_{n+2}) \\ &+ \dots + G(b_{m-1}, b_m, b_m) \\ &\leq (k^n + k^{n+1} + \dots + k^{m-1})G(b_0, b_1, b_1) \\ &\leq \frac{k^n}{1-k} G(b_0, b_1, b_1) \end{aligned}$$

taking limit as $n, m \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} G(b_n, b_m, b_m) = 0$. Thus $\{b_n\}$ is a G-Cauchy sequence in X.

Possibility2 If

$$\max\{G(b_{n-1}, b_n, b_n), G(b_{n-1}, b_{n+1}, b_{n+1}), G(b_n, b_{n+1}, b_{n+1})\} = G(b_{n-1}, b_{n+1}, b_{n+1})$$

from (3) and using rectangle inequality, we get

$$\begin{aligned} G(b_n, b_{n+1}, b_{n+1}) &\leq k G(b_{n-1}, b_{n+1}, b_{n+1}) \\ &\leq k(G(b_{n-1}, b_n, b_n) + G(b_n, b_{n+1}, b_{n+1})) \end{aligned}$$

this gives

$$G(b_n, b_{n+1}, b_{n+1}) \leq \frac{k}{1-k} G(b_{n-1}, b_n, b_n)$$

i.e.

$$G(b_n, b_{n+1}, b_{n+1}) \leq \beta G(b_{n-1}, b_n, b_n)$$

, where $\beta = \frac{k}{1-k}$ and $\beta < 1$ as $0 \leq k < \frac{1}{4}$ using possibility (1), we have $\{b_n\}$ is a G-Cauchy sequence in X.

Possibility3 If

$$\max\{G(b_{n-1}, b_n, b_n), G(b_{n-1}, b_{n+1}, b_{n+1}), G(b_n, b_{n+1}, b_{n+1})\} = G(b_n, b_{n+1}, b_{n+1})$$

then from (3), we have

$$G(b_n, b_{n+1}, b_{n+1}) \leq k G(b_n, b_{n+1}, b_{n+1})$$

which is a contradiction, since $k < \frac{1}{4}$. Therefore in all cases the sequence $\{b_n\}$ is a G-Cauchy sequence in X. Since (X, G) is G-complete Metric space. $\therefore$, there exists a point $x \in X$ s.t. $\lim_{n \rightarrow \infty} b_n = x$, we have

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} Ta_n = \lim_{n \rightarrow \infty} Sa_{n+1} = x$$

. since one of the maps T or S is continuous. Suppose we can assume that S is continuous, therefore $\lim_{n \rightarrow \infty} STa_n = \lim_{n \rightarrow \infty} SSa_n = Sx$ Also given that T and S are compatible. Therefore, $\lim_{n \rightarrow \infty} G(STa_n, TSa_n, TSa_n) = 0$ gives $\lim_{n \rightarrow \infty} TSa_n = Sx$, Now we claim that $Sx = x$, from (3) we have

$$(3.2) \quad G(TSa_n, Ta_n, Ta_n) \leq k \max \left\{ \begin{array}{l} G(STa_n, Sa_n, Sa_n), G(SSa_n, TSa_n, TSa_n), G(SSa_n, Ta_n, Ta_n) \\ G(SSa_n, Ta_n, Ta_n), G(Sa_n, Ta_n, Ta_n), G(Sa_n, TSa_n, TSa_n) \\ G(Sa_n, Ta_n, Ta_n), G(Sa_n, Ta_n, Ta_n), G(Sa_n, TSa_n, TSa_n) \\ G(Sa_n, Ta_n, Ta_n) \end{array} \right\}$$

taking, limit as $n \rightarrow \infty$ and by using proposition (2.11), we have

$$\begin{aligned} G(Sx, x, x) &\leq k \max\{G(Sx, x, x), G(x, Sx, Sx)\} \\ &\leq k \max\{G(Sx, x, x), 2G(Sx, x, x)\} \\ &= 2kG(Sx, x, x) \end{aligned}$$

which is a contradiction since $k < \frac{1}{4}$. Hence $Sx = x$, similarly we will show that $Sx = Tx = x$, for that we put $a = a_n, b = c = x$ in (3), we get

$$(3.3) \quad G(Ta_n, Tx, Tx) \leq k \max \left\{ \begin{array}{l} G(Sa_n, Sx, Sx), G(Sa_n, Ta_n, Ta_n), G(Sa_n, Tx, Tx) \\ G(Sa_n, Tx, Tx), G(Sx, Tx, Tx), G(Sx, Ta_n, Ta_n) \\ G(Sx, Tx, Tx), G(Sx, Tx, Tx), G(Sx, Ta_n, Ta_n) \\ G(Sx, Tx, Tx) \end{array} \right\}$$

taking limit as $n \rightarrow \infty$, we get

$$G(x, Tx, Tx) \leq kG(x, Tx, Tx)$$

, which is a contradiction since $k < \frac{1}{4}$. Hence $Tx = Sx = x$. Thus x is a common fixed point of T and S .

To prove Uniqueness, we assume that $x_1 \neq x$ be another common fixed point of T and S . Then $G(x_1, x, x) > 0$

(3.4)

$$G(x_1, x, x) = G(Tx_1, Tx, Tx) \leq k \max \left\{ \begin{array}{l} G(Sx_1, Sx, Sx), G(Sx_1, Tx_1, Tx_1), G(Sx_1, Tx, Tx) \\ G(Sx_1, Tx, Tx), G(Sx, Tx, Tx), G(Sx, Tx_1, Tx_1) \\ G(Sx, Tx, Tx), G(Sx, Tx, Tx), G(Sx, Tx_1, Tx_1) \\ G(Sx, Tx, Tx) \end{array} \right\}$$

By proposition (2.11), we get

$$\begin{aligned} G(x_1, x, x) &\leq k \max\{G(x_1, x, x), G(x, x_1, x_1)\} \\ &\leq k \max\{G(x_1, x, x), 2G(x_1, x, x)\} \\ &= 2kG(x_1, x, x), \end{aligned}$$

which is a contradiction since $k < \frac{1}{4}$ □

Example 3.2. Let $X = [-1, 1]$ and let (X, G) be a G -metric on X . G -metric function is defined as $G(a_1, b_1, c_1) = (|a_1 - b_1| + |b_1 - c_1| + |c_1 - a_1|)$, for all $a_1, b_1, c_1 \in X$. Then (X, G) be a G -metric space and define $Sx = \frac{x}{9}$ and $Tx = x$, then $S(X) \subseteq T(X)$. Also inequality (3) satisfies for all $a_1, b_1, c_1 \in X$. and 0 is the unique common fixed point of S and T .

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